2011

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. a) (Chain rule) Let $\omega(t)$ be a composite function

$$\omega(t) = f(X(t),\, Y(t)).$$

Write down the formula for $\omega'(t)$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $X'(t)$, $Y'(t)$.

Let $x_0 = X(t_0)$, $y_0 = Y(t_0)$. Then

$$\omega'(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\, X'(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\, Y'(t_0)$$

b) Derive the formula for the chain rule of Part a).
[Hint: You may use that, near $x = x_0, y = y_0$,

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0),$$

and linear Taylor's expansion of $X(t), Y(t)$ near $t = t_0$.]

We have

$$\omega(t) = f(X(t), Y(t)) \approx \frac{\partial f}{\partial x}(x_0,y_0)(X(t)-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(Y(t)-y_0).$$

Also,

$$X(t) \approx x_0 + X'(t_0)(t-t_0), \quad Y(t) \approx y_0 + Y'(t_0)(t-t_0).$$

Thus,

$$\frac{\omega(t)-\omega(t_0)}{t-t_0} \approx \frac{\partial f}{\partial x}(x_0,y_0)X'(t_0) + \frac{\partial f}{\partial y}(x_0,y_0)Y'(t_0).$$

When, $t \to t_0$ we obtain an equality for $\omega'(t_0)$.

c) Let R be a region on the xy plane defined by

$$x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0, y \le x$$

Find the integral

$$\iint_R e^{(x^2+y^2)} y^2\, dx dy.$$

Use polar coordinates, r, ϕ. Then R is defined by $0 \le r \le 2$, $0 \le \phi \le \pi/4$. Thus,

$$\iint_R e^{(x^2+y^2)} y^2 dxdy = \int_0^2 \int_0^{\pi/4} \exp(r^2)\, r^3 \sin^2(\phi)\, drd\phi$$

Since

$$\int_0^2 e^{r^2} r^3 dr = \frac{1}{2}[ue^u - e^u]_0^4 = \frac{1}{2}(3e^4+1),$$

$$\int_0^{\pi/4} \sin^2(\phi) d\phi == \frac{1}{2}[\phi - \frac{1}{2}\sin(2\phi)]_0^{\pi/4} = \frac{1}{4}(\frac{\pi}{2} - 1),$$

$$\iint_R e^{(x^2+y^2)} y^2 dxdy = \frac{1}{8}(3e^4+1)(\frac{\pi}{2} - 1).$$

2. a) Let $\mathbf{u} = (u_1, u_2, u_3)$ be a unit vector, $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1$. Let $f(x, y, z)$ be a function of 3 variables.

Define the directional derivative $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0, z_0)$.

$$\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0, z_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0)u_1 + \frac{\partial f}{\partial y}(x_0, y_0, z_0)u_2 + \frac{\partial f}{\partial z}(x_0, y_0, z_0)u_3$$

$$= (\nabla f(x_0, y_0, z_0),\, \mathbf{u}).$$

b) Show that $\frac{\partial f}{\partial \mathbf{u}}(x_0, y_0, z_0)$ achieves its maximum, as a function of $\mathbf{u}$, $|\mathbf{u}| = 1$, if

$$\mathbf{u} = \frac{\nabla f(x_0, y_0, z_0)}{|\nabla f(x_0, y_0, z_0)|}$$

As

$$(\mathbf{a}, \mathbf{b}) = |\mathbf{a}|\,|\mathbf{b}| \cos(\alpha),$$

where α is the angle between $\mathbf{a}$ and $\mathbf{b}$, when $|\mathbf{a}|, |\mathbf{b}|$ are fixed, $(\mathbf{a}, \mathbf{b})$ is maximal when $\alpha = 0$.

This defines $\mathbf{u}$.

c) Let the surface S be given as the graph of the function $f(x, y) = 1 + x^2 + y^2$, where (x, y) satisfy

$$x \ge 0, \quad y \ge 0, \quad x + y \le 1.$$

Find the surface integral

$$\iint_S \frac{e^{2x+y}}{\sqrt{4z-3}} dS.$$

If S is the graph of a function f over R, then

$$\iint_S g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dxdy.$$

Since R is given by $\{0 \le x \le 1,\ 0 \le y \le 1-x\}$ and $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$,

$$\iint_S \frac{\exp(2x+y)}{\sqrt{4z-3}}\,dS = \int_0^1 \left(\int_0^{1-x} e^{2x+y}dy\right)dx$$

$$= \int_0^1 \left(e^{x+1} - e^{2x}\right)dx = \frac{1}{2}(e^2+1) - e.$$

3. a) State the Divergence Theorem carefully.

Let D be a bounded domain in $\mathcal{R}^3$ surrounded by a smooth surface S. Let

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k},$$

be a smooth vector-field in D. Then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dxdydz.$$

b) Verify the Divergence Theorem when the vector-field $\mathbf{F}$ has the form,

$$\mathbf{F}(x,y,z) = a(xy^2\mathbf{i} + yx^2\mathbf{j}) + x^2\cos(\pi z)\mathbf{k}, \quad a > 0,$$

and D is the cylinder,

$$x^2+y^2 \le 1, \quad 0 \le z \le 1/2.$$

(i)

$$S = \Sigma \cup S_+ \cup S_-.$$

Here Σ is the cylindrical surface

$$\Sigma = \{(x,y,z) : x^2+y^2 = 1,\ 0 \le z \le 1/2\},$$

$S_+ = \{(x,y,z) : x^2+y^2 \le 1,\ z = 1/2\}$, $S_+ = \{(x,y,z) : x^2+y^2 \le 1,\ z = 0\}$,

On Σ,

$$\mathbf{n} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}} = x\mathbf{i}+y\mathbf{j},$$

so that, using cylindrical coordinates,

$$\mathbf{F}\cdot\mathbf{n} = 2ax^2y^2 = 2a\cos^2(\phi)\sin^2(\phi)$$

Thus,

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = 2a\int_0^{1/2}\int_0^{2\pi}\cos^2(\phi)\sin^2(\phi)\,d\phi dz$$

$$= \frac{a}{4}\int_0^{2\pi}\sin^2(2\phi)\,d\phi = \frac{\pi a}{4}$$

On S_+, $\mathbf{n} = \mathbf{k}$ and

$$\mathbf{F}\cdot\mathbf{n} = x^2\cos(\pi/2) = 0.$$

On S_-, $\mathbf{n} = -\mathbf{k}$ and

$$\mathbf{F}\cdot\mathbf{n} = -x^2,$$

and

$$\iint_{S_-}\mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{x^2+y^2\le 1} x^2\,dxdy =$$

$$-\int_0^1 r^3dr\int_0^{2\pi}\cos^2(\phi)\,d\phi = \frac{\pi}{4}.$$

(ii)

$$\nabla\cdot\mathbf{F} = a(y^2+x^2) - \pi x^2\sin(\pi z).$$

Thus,

$$\iiint_D \nabla\cdot\mathbf{F}dxdydz = \iiint_D \left(a(x^2+y^2) - \pi x^2\sin(\pi z)\right)\,dxdydz.$$

Using cylindrical coordinates, ρ, z, ϕ, we get

$$\iiint_D \nabla\cdot\mathbf{F}dxdydz = \int_0^1\int_0^{2\pi}\int_0^{1/2}\left[a\rho^2 - \pi\rho^2\cos^2(\phi)\sin(\pi z)\right]\rho\,d\rho dz d\phi$$

$$= \frac{\pi a}{4} + \frac{\pi}{4}\cos\pi z|_0^{1/2} = \frac{a-1}{4}\pi.$$

4. a. State Stoke's Theorem carefully.

Given a curve C with the anti-clockwise direction and a capping surface S, so that C is the boundary of S, let $\mathbf{F}(x,y,z)$ be a smooth vector field defined on S. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS.$$

b) Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x,y,z) = xy\mathbf{i} - z\mathbf{j} + y^2\mathbf{k}$$

and the surface S defined by

$$z - (x^2 + y^2) = 3, \quad z \leq 4.$$

i. The contour C is the circle $x^2 + y^2 = 1$ lying on the plane $z = 4$. Use parametrization

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 4\mathbf{k}, \quad \mathbf{r}'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}.$$

Thus,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\cos(t)\sin^2(t) - 4\cos(t),$$

so that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\cos(t)\sin^2(t) - 4\cos(t))\, d\phi = 0.$$

ii. On the other hand,

$$\nabla \times \mathbf{F} = (2y+1)\mathbf{i} - x\mathbf{k},$$

and S is the graph of the function $f(x,y) = 3 + (x^2+y^2)$ with the domain $R = \{(x,y) : x^2 + y^2 \leq 1\}$.
As

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y,$$

we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{x^2+y^2\leq 1} (-(2y+1)2x - x)\, dxdy$$

$$= \iint_{x^2+y^2\leq 1} (-4xy - 3x) dxdy = \int_0^1 \int_0^{2\pi} \left(-4r^2\cos(\phi)\sin(\phi) - 3r\cos(\phi)\right) r dr d\phi = 0.$$

c)) Let R be a triangle on the $xy-$plane with vorteces at the points

$$\mathbf{O} = (0,0),\ \mathbf{A} = (a,0),\ \mathbf{B} = (0,b), \quad a,b > 0.$$

Let $\mathbf{F}$ be a vector field,

$$\mathbf{F}(x,y) = \left(\sin(y)e^{x\sin(y)} + y\right)\mathbf{i} + x\cos(y)e^{x\sin(y)}\mathbf{j}.$$

Find

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the boundary of R traversed in the anti-cloakwise direction.

[Hint: You may use Green's Theorem.]

By Green's Theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = -\iint_R dxdy = -\frac{1}{2}ab.$$

5. a. Let $D = \mathcal{R}^3 \setminus \{(x,y,z) : x = y = 0\}$. Let C be a smooth curve in D. Let $\mathbf{F}(x,y,z)$ be a smooth vector field in D such that

$$\nabla\times\mathbf{F} = \mathbf{0}, \quad \text{in } \mathbf{D}.$$

Does it guarantee that $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the initial, $\mathbf{x_0}$, and terminal, $\mathbf{x_1}$, points of C? Expain your answer.
(Note that $\mathcal{R}^3 \setminus \{(x,y,z) : x = y = 0\}$ is the whole $\mathcal{R}^3$ without the $z-$axis).

Since D is not simply connected, the fact that $\nabla\times\mathbf{F} = \mathbf{0}$ does not guarantee that $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on $\mathbf{X_0}$ and $\mathbf{X_1}$.

b) Let the vector-field $\mathbf{F}(\mathbf{x})$, $\mathbf{x}\in\mathcal{R}^3$, be given by the formula

$$\mathbf{F}(x,y,z) = z\cos(y)e^{xz\cos(y)}\mathbf{i}$$

$$+\left(1 - xz\sin(y)e^{xz\cos(y)}\right)\mathbf{j} + x\cos(y)e^{xz\cos(y)}\mathbf{k}.$$

Is $\mathbf{F}$ a gradient vector-field and, if so, what is a corresponding potential?

If potential $f(x,y,z)$ exists, then

$$\frac{\partial f}{\partial x}(x,y,z) = z\cos(\phi)e^{xz\cos(y)},$$

so that

$$f(x,y,z) = e^{xz\cos(y)} + g(y,z).$$

Then,

$$\frac{\partial f}{\partial y}(x,y,z) = -xz\sin(y)e^{xz\cos(y)} + \frac{\partial g}{\partial y},$$

so that

$$\frac{\partial g}{\partial y} = 1, \quad f(x,y,z) = e^{xz\cos(y)} + y + h(z).$$

At last,

$$\frac{\partial f}{\partial z}(x,y,z) = x\cos(\phi)e^{xz\cos(y)} + h'(z),$$

so that

$$h(z) = C, \quad f(x,y,z) = e^{xz\cos(y)} + y + C.$$

Therefore, indeed,

$$\mathbf{F} = \nabla f,$$

and, therefore, $\nabla \times \mathbf{F} = \mathbf{0}$

c) Let

$$\mathbf{G} = z\cos(y)e^{xz\cos(y)}\mathbf{i}$$
$$+ \left(1 + xy - xz\sin(y)e^{xz\cos(y)}\right)\mathbf{j} + x\cos(y)e^{xz\cos(y)}\mathbf{k}.$$

Let C be a curve of the form

$$x = \cos(\pi t),\ y = sin(\pi t),\ z = t(1-t), \quad 0 \le t \le 1,$$

which connects the point $\mathbf{x}_0 = (1,0,0)$ with the point $\mathbf{x}_1 = (-1,0,0)$. Find

$$\int_C \mathbf{G} \cdot d\mathbf{r}.$$

[Hint: You may use part b).]

As $\mathbf{G} = \mathbf{F} + xy\mathbf{j}$, due to Part (b),

$$\int_C \mathbf{G} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C xy\mathbf{j} \cdot d\mathbf{r}.$$

We have

$$f(\mathbf{X}_1) - f(\mathbf{X}_0) = 0,$$

$$\int_C xy\mathbf{j} \cdot d\mathbf{r} = \int_0^1 \pi\cos^2(\pi t)\sin(\pi t)dt = \int_0^1 s^2 ds = \frac{2}{3}.$$

Thus,

$$\int_C \mathbf{G} \cdot d\mathbf{r} == \frac{2}{3}$$

6. a. State Fourier's Theorem carefully.

):

Let $f(x)$ be a $2\pi-$periodic piecewise continuously differentiable function. Then $f(x)$ can be represented by trigonometric Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n\cos(nx) + b_n\sin(nx)\right),$$

where a_0, a_n, b_n, $n = 1, 2, \ldots$ are its Fourier coefficients. The Fourier series at x converges to $f(x)$, if x is a point of continuity of f. If $f(x)$ has a jump in x, then the Fourier series at x converges to $\frac{1}{2}\left(f(x-0) + f(x+0)\right)$, where

$$f(x \pm 0) = \lim_{\epsilon \to +0} f(x \pm \epsilon).$$

b) Find the Fourier coefficients of the function $f(x)$ which is equal to

- π for $-\pi \le x \le 0$;
- $\pi - x$ for $0 \le x \le \pi$;
- continued 2π periodically from $(-\pi, \pi)$ to the whole $\mathcal{R}$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(2\pi^2 - \frac{1}{2}x^2|_0^{\pi}\right) = \frac{3}{4}\pi;$$

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \pi\cos(nx)dx - \int_0^{\pi} x\cos(nx)dx\right) = 0, \text{ if } n = 2k,$$

$$a_n = \frac{2}{\pi(2k-1)^2}, \text{ if } n = 2k-1, \quad k = 1, 2, \ldots;$$

$$b_n = \frac{1}{\pi}\left(\int_{-\pi}^{\pi} \pi\sin(nx)dx - \int_0^{\pi} x\sin(nx)dx\right) = \frac{(-1)^n}{n}.$$

c) Using Part 6 b) or otherwise, show that

$$\sum_{k=1}^{\infty} \frac{8}{(2k-1)^2} = \pi^2.$$

At $x = 0$, we get

$$\pi = \frac{3}{4}\pi + \sum_{k=1}^{\infty} \frac{2}{\pi(2k-1)^2}.$$

Rearanging this equation we get the result